The Assignment Problem

Although the assignment problem can be solved as an ordinary transportation problem or as a linear programming problem, its special structure can be exploited, resulting in a special-purpose algorithm, the so-called Hungarian Method. The method is due to H.W. Kuhn in 1955, named because it is based on work by the Hungarian mathematicians E. Egerváry and D. König; H.W. Kuhn himself also hails from Hungary. Recall that the assignment problem can be formulated as

\[
P: \text{Min } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}
\]

s.t. \( \sum_{j=1}^{n} x_{ij} = 1, i=1, \ldots, n \)

\( \sum_{i=1}^{n} x_{ij} = 1, j=1, \ldots, n \)

\( x_{ij} \geq 0, i, j = 1, \ldots, n. \)

We assume that all costs \( c_{ij} \) are nonnegative. (In case this condition is not satisfied, it is sufficient to add a sufficiently large positive constant to all cost parameters.) One can show that there will always exist at least one optimal solution with \( x_{ij} = 0 \) or 1 for all \( i \) and \( j \). The Hungarian Method uses this result, and considers as feasible solutions those that pick exactly one element \((i, j)\) in each row of the cost matrix \( C = (c_{ij}) \) (this is the element for which \( x_{ij} = 1 \)), and exactly one element \((i, j)\) in each column of the cost matrix \( C \) (again, for which \( x_{ij} = 1 \)). The method then proceeds in two phases. In the first phase, frequently referred to as Flood’s technique, we make use of the fact that if some arbitrary constant is added to or subtracted from each element in a row or column, then the total cost for all \( n! \) feasible assignments will be increased or reduced by the value of that constant. To demonstrate Flood’s technique, consider the following

**Example 1:** Given the cost matrix

\[
C = \begin{bmatrix}
10 & 9 & 7 & 8 \\
5 & 8 & 7 & 6 \\
5 & 4 & 6 & 7 \\
2 & 3 & 4 & 5
\end{bmatrix},
\]

for which we wish to find a minimal-cost assignment. In other words, pick four elements from this \([4 \times 4]\)-dimensional matrix, such that each row/column contributes exactly one element, and the sum of the four elements is minimized. We will work with nonnegative elements and will repeatedly modify the matrix in such a way that, if possible, a zero-sum assignment is found. Such an assignment must then be optimal. It follows that the corresponding assignment made in the original cost matrix \( C \) must also be optimal. Consider the first row and find its smallest element. In our example we find \( c_{13} = 7 \). The smallest elements in the other rows are \( c_{21} = 5 \), \( c_{32} = 4 \), and \( c_{41} = 2 \). Subtracting these elements from all elements in their respective rows, we obtain the revised matrix
This procedure always results in a matrix with at least one zero element in each row. We now repeat the process by starting with the revised matrix $C'$, determine the minimal in its columns and subtract them from their respective column elements. This will result in another revised matrix $C''$. In our example, there is a zero element in each of the first three columns (which is clearly the minimum), the column minima are 0, 0, 0, and 1. Subtracting them from the elements in the respective columns results in the matrix

$$
C'' = \begin{bmatrix}
3 & 2 & 0 & 0 \\
0 & 3 & 2 & 0 \\
1 & 0 & 2 & 2 \\
0 & 1 & 2 & 2
\end{bmatrix}
$$

At this point, Flood’s technique terminates and we attempt to find a set of zeroes, such that there is exactly one of the chosen elements in each row and one in each column. If this is possible, then we have found an optimal solution. In our example, this is indeed possible and the solution is shown by the circled elements in the matrix $C''$. In terms of the formulation, this means that $x_{13} = x_{24} = x_{32} = x_{41} = 1$, and $x_{ij} = 0$ for all other variables, so that the objective value equals $z = c_{13}x_{13} + c_{24}x_{24} + c_{32}x_{32} + c_{41}x_{41} = 7 + 6 + 4 + 2 = 19$.

Consider now a second example, in which it is not possible to find a solution right after applying Flood’s technique.

**Example 2:** Find a minimal cost assignment in the cost matrix

$$
C = \begin{bmatrix}
3 & 5 & 7 & 1 \\
9 & 8 & 12 & 10 \\
13 & 8 & 14 & 2 \\
5 & 7 & 10 & 6
\end{bmatrix}
$$

Subtraction of the row minima results in the revised matrix

$$
C' = \begin{bmatrix}
2 & 4 & 6 & 0 \\
1 & 0 & 4 & 2 \\
11 & 6 & 12 & 0 \\
0 & 2 & 5 & 1
\end{bmatrix}
$$

Subtraction of the column minima then results in the revised matrix
At this point, we attempt to find a set of assignments, one in each row and one in each column. This is, however, not possible. (Notice that the first and third rows in the matrix \( C'' \) both have a zero element in the last column, so that if we make one assignment, we cannot make the other as well, so that no assignment is possible that uses on the zero elements in the matrix).

This is when the Hungarian Method proceeds into its second phase. In this phase, we attempt to cover all zero elements by the smallest number of horizontal and/or vertical lines through the rows and columns. Here, we will do this by simple inspection, whereas a formal method is described in books such as Eiselt et al., 1987). In our example, three lines are needed. Two possibilities are shown below.

Here, we arbitrarily take the first choice. We now determine the smallest element in the matrix that is not covered at all. This element is then

- subtracted from all elements of the matrix that are not covered, and
- added to all element in the matrix that are covered twice.

In our example, the smallest element is “2” and after adding this value to and subtracting it from the appropriate elements, we obtain the matrix

\[
\begin{bmatrix}
0 & 2 & 0 & 0 \\
1 & 0 & 4 \\
9 & 4 & 6 & 0 \\
0 & 2 & 1 & 3
\end{bmatrix}
\]

in which we can find exactly one assignment on the zero elements. (We do not have a choice in the third and the fourth rows, then deleting all rows and columns in which we have made assignments, leaves only one possible assignment in the remaining rows). The assignment is shown by the circles and its objective value equals \( \bar{z} = 7 + 8 + 2 + 5 = 22 \).

By coincidence, if we had applied column reduction first and then row reduction in Flood’s technique, then the reduced matrix would have been
and we would have found the same solution right away.

For practice, we will also pursue the second option after covering the rows and columns of the matrix with lines. The smallest uncovered element is this case equals “1,” and subtracting one from all uncovered elements and adding it to those that are covered twice leads to the revised matrix

\[
\begin{bmatrix}
2 & 3 & 1 & 0 \\
2 & 0 & 0 & 4 \\
10 & 4 & 6 & 0 \\
0 & 1 & 0 & 2
\end{bmatrix}
\]

which does not have an assignment on the “0” elements. Hence we draw the lines again and again determine the smallest uncovered element. In this case, it is “1,” so that the next matrix is as shown below. It is apparent that there is an assignment in that matrix as is shown. It happens to be the exact same assignment we found earlier.

**Modifications**

The problem of finding a maximal rather than a minimal cost assignment can easily be handled by finding any element larger than the maximal element in the matrix, i.e., \( \omega \geq \max_{i,j} c_{ij} \). We then set up a pseudo-cost matrix \( \hat{C} = (\hat{c}_{ij}) \) with \( \hat{c}_{ij} = \omega - c_{ij} \). The Hungarian Method is then applied to the pseudo-cost matrix \( \hat{C} \), where a minimal-cost assignment corresponds to a maximal-cost assignment in the original cost matrix \( C \).

**Example 3:** Using the cost matrix from Example 1, i.e.,

\[
C = \begin{bmatrix}
10 & 9 & 7 & 8 \\
5 & 8 & 7 & 6 \\
5 & 4 & 6 & 7 \\
2 & 3 & 4 & 5
\end{bmatrix}
\]

we find \( \omega \geq 10 \). With a value of 10, we obtain
We now seek a minimal-cost assignment for $\hat{C}$, which is the same as finding a maximal-profit assignment for $C$. After performing the usual row- and column reduction, we find the revised matrix

$$
\begin{bmatrix}
0 & 1 & 2 & 2 \\
3 & 0 & 0 & 2 \\
2 & 3 & 0 & 0 \\
3 & 2 & 0 & 0 \\
\end{bmatrix}
$$

In this matrix we can find two alternative optimal solution that are shown below.

$$
\begin{bmatrix}
\text{0} & 1 & 2 & 2 \\
3 & \text{0} & 0 & 2 \\
2 & 3 & \text{0} & 0 \\
3 & 2 & 0 & \text{0} \\
\end{bmatrix} \quad \begin{bmatrix}
\text{0} & 1 & 2 & 2 \\
3 & \text{0} & 0 & 2 \\
2 & 3 & \text{0} & \text{0} \\
3 & 2 & \text{0} & 0 \\
\end{bmatrix}
$$

The objective value at optimum $z$ is determined by adding up the assigned elements in the original profit matrix $C$. We obtain $\bar{z} = c_{11} + c_{22} + c_{33} + c_{44} = 10 + 8 + 6 + 5 = 29$ in the first solution and $\bar{z} = c_{11} + c_{22} + c_{34} + c_{43} = 10 + 8 + 7 + 4 = 29$ for the second solution.

Since the unique cost-minimal assignment had a total cost of 19 (see Example 1), and the above two profit-maximal assignments with the same matrix $C$ had an objective value of 29, it follows that all other feasible assignments (of which there are $n! - 3 = 4! - 3 = 24 - 3 = 21$) must have objective values between these two bounds.

Finally, consider the situation, in which the assignment problem is unbalanced, i.e., when the cost matrix is not square.

*Example 4:* Consider the processing time matrix in Table 4.7 on page 152 and consider the figures as cost data:

$$
\begin{bmatrix}
5 & 1 & 9 & 4 & 9 \\
4 & 3 & 8 & 3 & 8 \\
7 & 5 & 6 & 4 & 7 \\
\end{bmatrix}
$$

The assignment problem must now be modified so as to read: “Pick exactly one element in each row in such a way that each column is used at most once and that the total sum of the three elements thus chosen is minimal.”

The Hungarian Method can be modified to treat this problem by adding two dummy rows, whose cost elements consist entirely of zeroes. In our numerical example, the resulting cost matrix is then
Since each column already includes multiple zeroes, only row reduction is necessary. We obtain the matrix below that does not allow us to choose an assignment entirely on the zero elements. Hence we have to cover all zeroes with horizontal and vertical lines.

The smallest uncovered element is now “1,” so that we obtain the new matrix

In this matrix we can find two alternative optimal solutions that are shown below.

Our unique optimal assignment consists of the elements shown above with the total cost of \( Z = 4 + 1 + 4 = 9 \). In terms of the workload balancing problem on p. 152, we faced the task of minimizing the assignment costs and at optimum, we assign task \( T_2 \) to worker \( W_1 \), task \( T_1 \) to worker \( W_2 \), and task \( T_4 \) to worker \( W_3 \). The tasks \( T_3 \) and \( T_5 \) remain unassigned. Inspecting the original data, this solution makes intuitive sense, as the tasks \( T_3 \) and \( T_5 \) both tasks are very time-consuming regardless with worker they are assigned to.